The Nyquist stability criterion

- Loop analysis
- Principle of variation of the argument
- The Nyquist criterion
- Summary

Material: Franklin - Powell Ch.6.3, Murray-Åström Ch.9, lecture notes

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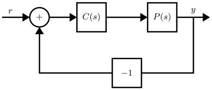
The closed-loop transfer function



$$P(s) = \frac{n_p(s)}{d_p(s)}, \quad C(s) = \frac{n_c(s)}{d_c(s)}$$

$$G_{yr} = \frac{PC}{1+PC} = \frac{n_p(s)n_c(s)}{d_p(s)d_c(s) + n_p(s)n_c(s)}$$

$$\lambda(s) = d_p(s)d_c(s) + n_p(s)n_c(s)$$

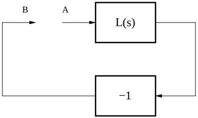
Stable if $Re(\lambda_i) \leq 0$

Pro: approach is straightforward

Con: not easy to tell how the controller should be modified to make an unstable system stable

Nyquist idea

Introduce the **open-loop transfer function**: $L(s) = P(s)C(s)$



Let a sinusoid of frequency ω_0 be injected at point A. In steady-state the signal at point B will also be a sinusoid with the frequency ω_0.

- What are the conditions for having a periodic oscillation in the loop? $L(j\omega_0) = -1$

Note One of the powerful concepts embedded in Nyquist's approach to stability analysis is that it allows us to determine stability of a closed-loop system by looking at property of the open-loop transfer function

A simple stability condition

$L(j\omega) = -1 \Rightarrow$ the Nyquist curve of $L(s)$ intersects -1 (critical point)

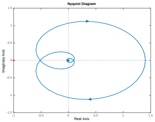


Figure: Nyquist plot of the transfer function $L(s) = 1.4e^{-s}/(s+1)^2$

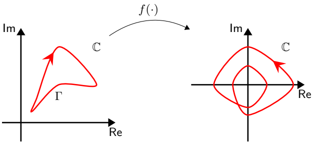
Intuitively, it seems reasonable that the closed-loop system would be stable if $\mid L(j\omega) \mid < 1$, i.e., the critical point -1 is on the left hand side of the Nyquist curve.

Contour evaluations

$$\Gamma : [a, b] \to \mathbb{C} \qquad\qquad f(\Gamma) : [a, b] \to \mathbb{C}$$

$$z \to \Gamma(z) \qquad\qquad z \to f(\Gamma(z))$$

$\Gamma(a) = \Gamma(b)$, clockwise $\qquad$ $f(\cdot)$ analytic, $f(\Gamma(a)) = f(\Gamma(b))$



Principle of variation of the argument

Theorem (variation of the argument) Let Γ be a simple closed contour in the complex plane and let D the region enclosed by the contour. Assume the function $f(\cdot)$ is analytic in D and on Γ except at a finite number of poles in D. Then, the variation Δ_Γ in the angle when z traverses Γ is:

$$N_w = \frac{1}{2\pi}\Delta_\Gamma \arg f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}dz = N - P$$

where N is the number of zeros of $f(\cdot)$ in D and P the number of poles of $f(\cdot)$ in D. N_w is called winding number.

Principle of variation of the argument

Proof (residue theorem)

- Assume $z = a$ a zero of multiplicity m.

In the neighborhood of $z = a$, $f(z) = (z - a)^m q(z)$, $q(z)$ analytic, $q(a) \neq 0$

$$\frac{f'(z)}{f(z)} = \frac{m}{z-a} + \frac{q'(z)}{q(z)} \qquad \lim_{z\to a}(z-a)\frac{f'(z)}{f(z)} = m$$

- Assume $z = a$ a pole of multiplicity m.

In the neighborhood of $z = a$, $f(z) = \frac{1}{(z-a)^m}p(z)$, $p(z)$ analytic, $p(a) \neq 0$

$$\frac{f'(z)}{f(z)} = \frac{-m}{z-a} + \frac{p'(z)}{p(z)} \qquad \lim_{z\to a}(z-a)\frac{f'(z)}{f(z)} = -m$$

Principle of variation of the argument (cont'd)

Proof (cont'd) For the residue theorem:

$$\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)}dz = \sum_{a\in Z(f)\cup P(f)} \text{res}\left(\frac{f'}{f}, a\right) = N - P.$$

$$\frac{f'(z)}{f(z)} = \frac{d}{dz}\log f(z)$$

$$\int_\Gamma \frac{f'(z)}{f(z)}dz = \int_\Gamma d\log f(z) = \Delta_\Gamma \log f(z)$$

$$\log f(z) = \log \mid f(z) \mid + i\arg f(z)$$

$$\Delta_\Gamma \log f(z) = \Delta_\Gamma \log \mid f(z) \mid + i\Delta_\Gamma \arg f(z)$$

$\Delta_\Gamma \log \mid f(z) \mid = 0$ since the variation around a closed contour is zero

Then: $\Delta_\Gamma \log f(z) = \Delta_\Gamma i \arg f(z) = 2\pi i N_w$

Application to stability analysis

Consider a closed-loop system with open-loop transfer function $L(s)$.

Apply the argument principle to $f(s) = 1 + L(s)$.

Choose the contour Γ s.t. the region D is the inner of a half circle in the RHP with center in the origin and radius R. If $f(\cdot)$ has poles on the imaginary axis, introduce small semicircles with radii r. By letting $R \to \infty$ and $r \to 0$ the Nyquist contour is obtained.

Notice that $1 + L(s)$ is simply $L(s)$ shifted by one, therefore if the plot of $1 + L(s)$ encircles the origin, the plot of $L(s)$ will encircle -1 on the real axis.

The Nyquist theorem

Theorem (Nyquist's stability criterion) Consider a closed-loop system with open-loop transfer function $L(s)$. The closed-loop poles of the system are the zeros of the function $f(s) = 1 + L(s)$. To find the number of zeros in the right half plane, we investigate the winding number of $f(s)$ as s moves along Γ in the clockwise direction. Then:

- N = number of unstable zeros of $1 + L(s)$
- P = number of unstable poles of $1 + L(s)$
- $\frac{1}{2\pi}\Delta \arg(1 + L(s)) = N_w = N - P$
- N_w = net number of clockwise encirclements of -1
- If $N = N_w + P = 0$ stable closed-loop system!!!

Example (Open-loop unstable system)

Consider the open-loop transfer function

$$L(s) = k\frac{(s+2)(s+1.9)}{s^3 - 1}$$

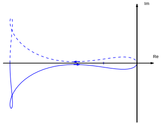
Is it possible to stabilize the closed-loop obtained with unit negative feedback by choosing k appropriately?

- $\omega \to 0^+$

  Re $[L(j\omega)] \to -3.8$ Im$[L(j\omega)] \to 0$
- $\omega \simeq +\infty$

  $L(j\omega) \to 0$

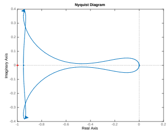
for $\omega > 0$ Im$[L(j\omega)] < 0$



Example (Open-loop unstable system)

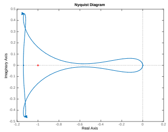
Nyquist criterion: $N = N_w + P$

$k < \frac{1}{3.8}$



$N_w = 0,\ P = 1,\ N = 1$

$k > \frac{1}{3.8}$



$N_w = -1,\ P = 1,\ N = 0$

Example

Consider the open-loop transfer function

$$L(s) = k\frac{(2s+1)}{s^2(s+1)(0.4s+1)}$$

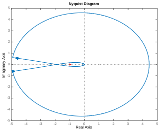
Is it possible to stabilize the closed-loop obtained with unit negative feedback by choosing k appropriately?

- $\omega \to 0^+$

  $\mathbb{R}[L(j\omega)] \to -\infty, \mathrm{Im}[L(j\omega)] \to -\infty$
- $\omega \simeq +\infty$

  $L(j\omega) \to 0$

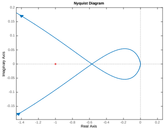
Intersection with real axis $\omega \simeq -1.9$



Example

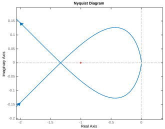
Nyquist criterion: $N = N_w + P$

$k < \frac{1}{1.9}$



$N_w = 0,\ P = 0,\ N = 0$

$k > \frac{1}{1.9}$



$N_w = 1,\ P = 0,\ N = 1$

The Nyquist stability criterion

- provides a necessary and sufficient condition for closed-loop stability based on the open-loop transfer function
- based on concepts from complex numbers analysis
- provides a measure for "how far" from instability the closed-loop system is (robustness analysis)
- useful in dealing with open-loop unstable systems, non minimum phase systems and systems with pure delay

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